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OSCILLATIONS OF A SOLID ABOUT THE PERMANENT STEADY-STATE ROTATIONS

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The method of averaging [1] is used to obtain the amplitudes and frequencies of the oscillations of a solid about the permanent, steady-state rotations, is the Euler, Lagrange and Kowalewska cases.

The method consists, essentially, of passing to the normal coordinates (amplitudes and angle variables) and replacing the nonlinear terms (which are small compared with the linear) in the equations of motion by their integral values averaged over the periods of the angle variables. The characteristic equation of the first order approximation has, in these cases, imaginary roots and one or two zero roots, therefore it is convenient to pass from x_i to a_k, ξ_k and u_k. This transformation is expressed for the characteristic matrix of the first approximation equations in the matrix form as follows:

$$x=\sum_i a_i \left[\operatorname{Re} V_1(i\omega_i)\cos u_i - \operatorname{Im} V_1(i\omega_i)\sin u_i\right] + V_2'(0)\,\xi_i \tag{1}$$

where V_1 and V_2 are the zero columns accompanying the pure imaginary and zero roots of the adjoint matrix.

We shall investigate the equations of perturbed motion of a solid with principal moments of inertia denoted by A, B and C, projections of the vector of angular velocity of rotation on the axes of inertia by p, q and r, and coordinates of the center of mass on the axes of inertia by x_0, y_0 and z_0. The body is acted upon by a homogeneous or central Newtonian force field

$$U = -mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3)$$
$$U = -mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) - \mu\,(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)\,/\,2$$

where γ_i are the direction cosines of the z-axis relative to the principal axes of inertia.

T h e E u l e r c a s e $(x_0 = y_0 = z_0 = 0)$. The equations of motion admit the particular solution $p = q = 0,\ r = \omega$. In the case of perturbed motion we assume that $p = x_1,\ q = x_2,\ r = \omega + x_3$, and obtain

$$x_1^{\cdot} = -a\omega x_2 - a x_2 x_3,\quad x_2 = b\omega x_1 + b x_1 x_3,\quad x_3^{\cdot} = c x_1 x_2$$
$$a = (C - B)\,/\,A,\quad b = (C - A)\,/\,B,\quad c = (A - B)\,/\,C$$

In the case of steady rotations about the z-axis $(ab > 0)$ the characteristic equation $\lambda\,(\lambda^2 + ab\omega^2) = 0$ has a zero root and a pair of purely imaginary roots $\lambda_1 = 0$, $\lambda_{23} = \pm i\omega\sqrt{ab}$, therefore the transformation (1) has the form

$$x_1 = -\alpha\sqrt{ab}\sin u, \quad x_2 = \alpha b\cos u, \quad x_3 = \xi$$

and the equations in terms of the normal coordinates become

$$\alpha^{\cdot} = 0, \ \xi^{\cdot} = -bc\sqrt{ab}\alpha^2 \sin u \cos u, \ u^{\cdot} = \sqrt{ab}\,(\omega + \xi) \tag{2}$$

Equations (2) averaged over u have the solution $\alpha = \alpha_0$, $\xi = \xi_0$, $u = \sqrt{ab}\,(\omega + \xi_0)\,t + u_0$. When $c = 0$ $(A = B)$, the above functions become a solution of the exact (not averaged) equations. The solution determines the oscillation of the body with respect to the variables x_1 and x_2, with period equal to $2\pi / \sqrt{ab}\,(\omega + \xi_0)$. The phase trajectories in the variable space (x_1, x_2, x_3) are ellipses which lie on the plane parallel to the x_1x_2-plane.

When the body moves in a central Newtonian force field, the Euler — Poisson equation admit the particular solution $p = q = 0$, $r = \omega$, $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = 1$ and the equation of perturbed motion are

$$\begin{aligned} &x_1^{\cdot} = -a\omega x_2 + \mu a y_2 - a\,(x_2x_3 - \mu y_2y_3) \\ &x_2^{\cdot} = b\omega x_1 - \mu b y_1 + b\,(x_1x_3 - \mu y_1y_3), \ x_3^{\cdot} = c\,(x_1x_2 - \mu y_1y_2) \\ &y_1^{\cdot} = -x_2 + \omega y_2 + x_3y_2 - x_2y_3 \\ &y_2^{\cdot} = x_1 - \omega y_1 + x_1y_3 - x_3y_1, \quad y_3^{\cdot} = x_2y_1 - x_1y_2 \end{aligned} \tag{3}$$

The characteristic equation of the linearized system (3)

$$\lambda^2\,(\lambda^4 + m\lambda^2 + n) = 0, \quad m = \omega^2\,(1 + ab) - \mu\,(a + b), \quad n = (\mu - \omega^2)^2 ab$$

has two zero roots and two pairs of purely imaginary roots $\pm i\omega_k$ $(k = 1,2;\ m = \omega_1^2 + \omega_2^2,\ n = \omega_1^2\omega_2^2)$, therefore the transformation (1) assumes in this case the form

$$\begin{aligned} x_1 = -a_i d_{1i}\sin u_i, \quad x_2 = a_i c_{1i}\cos u_i, \ x_3 = -n\xi_1 \\ y_1 = -a_i d_{2i}\sin u_i, \ y_2 = a_i c_{2i}\cos u_i, \ y_3 = -n\xi_2 \end{aligned} \tag{4}$$

where the summation over i is carried out from one to two, and

$$\begin{aligned} c_{1k} &= -b\omega\omega_k^2\,(\omega_k^2 + \mu - \omega^2), \ c_{2k} = -\omega_k^2\,(\omega_k^2 + \mu b - b\omega^2) \\ d_{1k} &= -\omega_k^3\,(\omega_k^2 + \mu b - \omega^2), \ d_{2k} = \omega\omega_k\,(1 - b), \ k = 1,2 \end{aligned}$$

After this, the equations (3) transform into the following equations in normal coordinates:

$$\begin{aligned} &a_k^{\cdot} = (\alpha_{ij}^{(k)}\sin u_k\cos u_j + \beta_{ij}^{(k)}\cos u_k\sin u_j)\xi_i a_j \\ &u_k^{\cdot} = \omega_k + (-1)^k\,(\alpha_{ij}^{(k)}\cos u_k\cos u_j - \beta_{ij}^{(k)}\sin u_k\sin u_j)\,a_k^{-1}\xi_i a_j \\ &n\xi_k^{\cdot} = \gamma_{ij}^{(k)} a_i a_j \sin u_i\cos u_j \\ &\alpha_{ij}^{(k)} = (-1)^{i-1} n\,(\mu^{i-1} a d_{2,1+k} c_{ij} + d_{1,1+k} c_{1+i,j})\,(d_{12}d_{21} - d_{11}d_{22})^{-1} \\ &\beta_{ij}^{(k)} = (-1)^{i-1} n\,(\mu^{i-1} b d_{ij} c_{2,1+k} + d_{1+i,j} c_{1,1+k})\,(c_{11}c_{22} - c_{12}c_{21})^{-1} \\ &\gamma_{ij}^{(k)} = c^{k-1}\,(c_{1j}d_{1+k,i} - \mu^{k-1} c_{2j}d_{ki}) \end{aligned} \tag{5}$$

Here and in what follows $k = 1,2$, and the indices i and j denote summation from one to two.

Averaging the right hand sides of (5) over the angle variables u_k, we obtain the abbreviated equations and their solutions

$$a_k^{\cdot} = 0,\ \xi_k^{\cdot} = 0,\ u_k^{\cdot} = \omega_k + {}^1/_2\,(-1)^k\,(\alpha_{ik}^{(k)} - \beta_{ik}^{(ik)})\,\xi_{i0} = \omega_k^* \qquad (6)$$
$$a_k = a_{k0},\quad \xi_k = \xi_{k0},\quad u_k = \omega_k^*\,t + u_{k0}$$

From (6) it follows that the variables (4) are quasi-periodic functions of time, with the periods equal to $T_k = 2\pi / \omega_k^*$.

L a g r a n g e c a s e $(x_0 = y_0 = 0,\ A = B)$. The Euler — Poisson equations admit the particular solution $p = q = \gamma_1 = \gamma_2 = 0,\ \gamma_3 = 1,\ r = \omega$.

Assuming in the perturbed motion that $p = x_1,\ q = x_2,\ \gamma_1 = y_1,\ \gamma_2 = y_2,\ \gamma_3 = 1 + y_3$, we obtain the equations of perturbed motion in a homogeneous gravitational field

$$x_1^{\cdot} = ax_2 + by_2,\quad x_2^{\cdot} = -ax_1 - by_1 \qquad (7)$$
$$y_1^{\cdot} = -x_2 + ry_2 - x_2y_3,\ y_2^{\cdot} = x_1 - ry_1 + x_1y_3,\ y_3^{\cdot} = x_2y_1 - x_1y_2$$
$$a = (A - C)r / A,\ b = mgz_0 / A,\ r^{\cdot} \equiv 0$$

The characteristic equation $\lambda\,[\lambda^4 + (r^2 + a - 2b)\,\lambda^2 + (b + ar)^2] = 0$ of the linearized system (7) has a single zero root and two pairs of purely imaginary roots $\pm i\omega_k,\ \omega_1^2 + \omega_2^2 = r^2 + a - 2b,\ \omega_1^2\omega_2^2 = (b + ar)^2$, therefore the transformation (1) to three amplitudes $\xi,\ a_1$ and a_2 and two angle variables u_1 and u_2, has the form

$$x_1 = -a_id_{1i}\sin u_i,\quad x_2 = a_ic_{1i}\cos u_i \qquad (8)$$
$$y_1 = -a_id_{2i}\sin u_i,\ \ y_2 = a_ic_{2i}\cos u_i,\ \ y_3 = \xi$$
$$c_{1k} = r\,(b + ar) - a\omega_k^2,\quad c_{2k} = \omega^2 + b + ar$$
$$d_{1k} = \omega_k\,(\omega_k^2 + b - r^2),\ d_{2k} = -\omega_k\,(a + r),\ k = 1,2$$

where the summation over i is carried out from one to two. The equations in normal coordinates now become

$$a_k^{\cdot} = (c_{1,1+k}d_{1i}\sin u_i\cos u_k - c_{1i}d_{1,1+k}\cos u_i\sin u_k)\xi a_i d^{-1} \qquad (9)$$
$$u_k^{\cdot} = \omega_k + (-1)^k\,(c_{1,1+k}d_{1i}\sin u_i\sin u_k + c_{2i}d_{1,1+k}\cos u_i\cos u_k)\,a_k^{-1}\xi a_i d^{-1}$$
$$\xi_k^{\cdot} = a_ia_j\,(c_{2i}d_{1j} - c_{1i}d_{2j})\sin u_j\cos u_i,\quad d = (a + r)\,(b + ar)\,(\omega_2^2 - \omega_1^2)$$

Averaging the right hand sides of (9) over the angle variables u_k, we obtain the abbreviated equations and their solutions in the form (6), where

$$\omega_k^* = \omega_k + (-1)^k\,(c_{1,\,1+k}d_{1k} + c_{2k}d_{1,\,1+k})\,\xi_0\,(2d)^{-1}$$

which denotes, in terms of the variables x_k, y_k, the quasi-periodic motions with periods $T_k = 2\pi / \omega_k^*$.

The equations of perturbed motion in a central Newtonian force field coincide, in the linear approximation, with (7), and can therefore be reduced by means of the transformation (8), to the equations in normal coordinates where $c_{1,1+k}d_{1i}$ are replaced by α_{ki}; $c_{1i}d_{1,1+k}$ by β_{ki}, and

$$\alpha_{ki} = c_{1,\,1+k}d_{1i} - \mu c_{2,\,1+k}d_{2i},\quad \beta_{ki} = c_{1i}d_{1,\,1+k} - \mu c_{2i}d_{2,\,1+k}$$

The abbreviated equations and their solutions have the form (6) where

$$\omega_k^* = \omega_k + (-1)^k\,(\alpha_{kk} + \beta_{kk})\xi_0(2d)^{-1}$$

This means that the body executes quasi-periodic oscillations with respect to the variables (8), with periods $T_k = 2\pi / \omega_k^*$.

K o w a l e w s k a c a s e ($y_0 = z_0 = 0,\ A = B = 2C$). The Euler — Poisson equations admit, in the case of a homogeneous or a central Newtonian force field, the particular solution $p = \omega,\ \gamma_1 = 1,\ q = r = \gamma_2 = \gamma_3 = 0$. The equations of perturbed motion of a heavy solid have the form

$$\begin{aligned} &2x_1^{\cdot} = x_2x_3, \quad 2x_2^{\cdot} = -\omega x_3 + ay_3 - x_1x_3 \\ &x_3^{\cdot} = -ay_2, \quad a = mgx_0 / C \\ &y_1^{\cdot} = x_3y_2 - x_2y_3, \quad y_2^{\cdot} = -x_3 + \omega y_3 + x_1y_3 - x_3y_1 \quad y_3^{\cdot} = x_2 - \omega y_2 + \\ &\quad x_2y_1 - x_1y_2 \end{aligned} \tag{10}$$

The characteristic equation $2\lambda^2 [2\lambda^4 + (2\omega^2 - 3a)\lambda^2 + a(a - \omega^2)] = 0$ of the linearized system (10) has, in the case of permanent, steady state rotations($a < 0$), two zero roots and two pairs of purely imaginary roots $\pm i\omega_k,\ 2(\omega_1^2 + \omega_2^2) = 2\omega^2 - 3a,\ 2\omega_1^2\omega_2^2 = a(a - \omega^2)$. We can therefore use the transformation

$$\begin{aligned} &x_1 = \xi_2,\ x_2 = -a_id_{1i}\sin u_i,\ x_3 = a_ic_{1i}\cos u_i \\ &y_1 = \xi_2,\ y_2 = -a_id_{2i}\sin u_i,\ y_3 = a_ic_{2i}\cos u_i \\ &c_{1k} = a\omega,\ c_{2k} = a + \omega_k^2,\ d_{1k} = \omega_k(\omega_k^2 + a - \omega^2),\ d_{2k} = -\omega\omega_k \end{aligned} \tag{11}$$

to reduce the equations (10) to the following equations in normal coordinates:

$$\begin{aligned} &a_k^{\cdot} = a_i(\alpha_{ki}\cos u_k \sin u_i - \beta_{ki}\sin u_k \cos u_i) \\ &u_k^{\cdot} = \omega_k + (-1)^k a_ia_{k+1}(\alpha_{ki}\sin u_k \sin u_i + \beta_{ki}\cos u_k \cos u_i)(a_1a_2)^{-1} \\ &\xi_1^{\cdot} = -2^{-1} a_ia_jd_{1j}c_{1i}\cos u_i \sin u_j,\quad \xi_2^{\cdot} = (d_{2i}\xi_1 - d_{1i}\xi_2)\, a_i \sin u_i \cdot \cos u_j \\ &\alpha_{ki} = a\omega(-d_{2i}\xi_1 + d_{1i}\xi_2)(c_{11}c_{22} - c_{12}c_{21})^{-1} \\ &\beta_{ki} = [-(2^{-1}d_{2,1+k}c_{1i} + d_{1,1+k}c_{2i})\,\xi_1 + d_{1,1+k}c_{1i}\xi_2](d_{11}d_{22} - d_{12}d_{21})^{-1} \end{aligned} \tag{12}$$

The abbreviated equations and their solutions have the form (6) where

$$\omega_k^* = \omega_k + (-1)^k 2^{-1}(\alpha_{kk} + \beta_{kk}) \tag{13}$$

i.e. the coordinates (11) are quasi-periodic functions of time with periods $T_k = 2\pi / \omega_k^*$.

If the solid moves in a central Newtonian force field, then the first two equations of perturbed motion (10) will become

$$2x_1^{\cdot} = x_2x_3 - \mu y_2y_3, \quad 2x_2^{\cdot} = -\omega x_3 + (a + \mu)y_3 - x_1x_3 + \mu y_1y_3$$

with the remaining equations of (10) unchanged.

The characteristic equation of the linearized equations of perturbed motion $2\lambda^2 [2\lambda^4 + (2\omega^2 - 3a - \mu)\lambda^2 + a(a + \mu - \omega^2)] = 0$ will now have two zero roots and two pairs of purely imaginary roots $\pm i\omega_k,\ 2(\omega_1^2 + \omega_2^2) = 2\omega^2 - 3a - \mu,\ 2\omega_1^2\omega_2^2 = a(a + \mu - \omega^2)$. Using the relations (11), we can transform these equations to the equations in normal coordinates (12) where the quantities d_{ki} and $\xi_1^{\cdot}$ increase by $2^{-1}\mu d_{2,1+k}c_{2i}\xi_2$ and $2^{-1} a_ia_jd_{2j}c_{2i}\cos u_i \sin u_j$ respectively. The abbreviated equations will assume the form (6) where ω_k^* is given by (13). This implies that the motion will be quasi-periodic in x_i, y_i, with two periods.

REFERENCES

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